

Miniversal Deformations of Marked Matrices

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Abstract

Given the set of square matrices $\mathcal{M} \subset M_{n+m}(C)$ that keep the subspace $W = C^n \times \{0\} \subset C^{n+m}$ invariant, we obtain the implicit form of a miniversal deformation of a matrix $a \in \mathcal{M}$, and we compute it explicitly when this matrix is marked (this is, if there is a permutation matrix $p \in M_{n+m}(C)$ such that $p^{-1}ap$ is a Jordan matrix). We derive some applications to tackle the classical Carlson problem.

1 Introduction

In [4] one proves that all the solutions to the Carlson problem appear in any neighbourhood of the simplest matrices, the so-called marked matrices. Studying the perturbations of this type of matrices is the central goal of this paper.

More precisely, we recall that the Carlson problem consists in obtaining the possible Jordan invariants of a matrix of the form

$$a = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

when those of A and B are prescribed. Notice that the 0 block means that the corresponding subspace is a -invariant, A and B being the matrices of the restriction to this invariant subspace and of the corresponding quotient map, respectively, in the associated basis.

As is well known one can assume that A, B are nilpotent Jordan matrices. Then, trivial solutions for the Segre characteristic of a are obtained by taking C in such a way that a is marked; that is to say, a becomes a Jordan matrix by conjugation with a permutation matrix. As we have said above, any other solution can be realized by perturbing marked matrices; therefore, each solution is represented in the versal deformations of some of them.

Versal deformation has been introduced by Arnold in [1] to study the variations of the invariants of a square matrix when its entries are perturbed, and thanks to a natural generalization contained in [12], the same technique has been applied to pairs, pencils, etc. ([5], [8],...). In [7] versal deformations of invariant subspaces with regard to a fixed endomorphism are described. Here the subspace is prescribed; thus we are interested in a local canonical form of the differentiable families of endomorphisms having this invariant subspace, or equivalently, of square matrices as a above.

In particular, we characterize a miniversal deformation of a matrix of the form a with regard to the changes of basis which preserve the 0 block, and we compute it explicitly when a is marked nilpotent. We then derive the referred applications to tackle the Carlson problem.

The organization of this paper is as follows.

In section (2), we obtain the implicit form of a miniversal deformation of the matrix a by applying Arnold's technique (2.10).

In section (3), we apply this theorem to obtain an explicit form of a first miniversal deformation of a marked matrix (3.8) and in (3.13), we obtain a second miniversal deformation without repeated parameters.

Finally, we study the relation between the obtained deformations and the Carlson problem in the last section. Particularly, the deformations that preserve the restriction A and the quotient B or, in other words, the pair of partitions (γ, β) of their Segre characteristics are the deformations with the only non zero parameters in the right upper block. We note that we obtain matricial realizations of all the compatible Littlewood-Richardson sequences with the pair (γ, β) of Segre characteristics among the deformations of a matricial realization of the Carlson compatible triple $(\gamma \cup \beta, \gamma, \beta)$ (4.4). By the union partition we mean the reordered union of two sets of partitions.

We denote by \mathcal{M} the set of matrices that preserves the subspace $C^n \times \{0\} \subset C^{n+m}$,

$$\mathcal{M} = \{a \in M_{n+m}(C) : a = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, A \in M_n(C), B \in M_m(C), C \in M_{n,m}(C)\}.$$

a^* will be the conjugate-transposed matrix of a and \mathcal{G} will be the group of invertible matrices of \mathcal{M} .

A *partition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}, 0, \dots)$ is a non increasing sequence of non negative integers

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\ell(\alpha)} > 0$$

where $\ell(\alpha)$ is its *length* and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)}$ its *weight*.

The *conjugate partition* $\alpha^* = (\alpha_1^*, \alpha_2^*, \dots)$ of the partition α is defined by

$$\alpha_j^* = \#\{1 \leq i \leq \ell(\alpha) : \alpha_i \geq j\}.$$

Notice that $\alpha_1^* = \ell(\alpha)$, $\ell(\alpha^*) = \alpha_1$, $|\alpha^*| = |\alpha|$, $(\alpha^*)^* = \alpha$.

Let α and β be two partitions. Then, the union partition $\alpha \cup \beta$ is the partition obtained by reordering the union of the two sets of partitions.

2 Miniversal Deformations

In order to study the perturbations of the numerical invariants of a square matrix with regard to the usual conjugation relation associated to changes of basis, Arnold introduces the so-called versal deformations in [1]. The starting point is the fact that the corresponding equivalence classes are orbits by the action of the linear group and, hence, they are submanifolds. Versal

deformations can then be obtained as submanifolds which are transverse to the orbit of the given matrix.

Arnold's techniques can be generalized to other situations where this basic fact holds. Let us see that this is our case.

Definition 2.1 *We consider the action of the group \mathcal{G} on the differentiable manifold \mathcal{M} defined by the conjugation*

$$\begin{aligned} \mathcal{G} \times \mathcal{M} &\longrightarrow \mathcal{M} \\ (p, a) &\longmapsto p * a = p^{-1}ap \end{aligned}$$

*The orbit of the matrix $a \in \mathcal{M}$, $\mathcal{O}_a = \{p * a : p \in \mathcal{G}\}$, is the equivalence class of $a \in \mathcal{M}$ with regard to the relation given by the group action.*

Definition 2.2 *Let \mathcal{V} be a manifold (for example, \mathcal{M} or \mathcal{G}). A deformation of $a \in \mathcal{V}$ is a differentiable map*

$$\varphi : \Lambda \longrightarrow \mathcal{V}$$

where Λ is a neighbourhood of the origin in C^l and $\varphi(0) = a$. We also say that the image $\varphi(\Lambda)$ is a family of deformations of the central element $a \in \mathcal{V}$.

The set Λ is called the basis of the deformation and l its dimension. We say that λ_i is a parameter of the deformation if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \Lambda$.

For example, every local parametrization of a submanifold $\mathcal{S} \subset \mathcal{V}$ where $a \in \mathcal{S}$ is a deformation of a . We will simply say that \mathcal{S} is a deformation of a .

A deformation is called “versal” if any other deformation is induced from it in the following sense:

Definition 2.3 *A deformation of $a \in \mathcal{M}$, $\varphi : \Lambda \longrightarrow \mathcal{M}$ ($\Lambda \subset C^r$) is called versal if, given another deformation of $a \in \mathcal{M}$, $\psi : \Gamma \longrightarrow \mathcal{M}$, there is a neighbourhood of the origin $\Gamma' \subset \Gamma$, a differentiable map $\rho : \Gamma' \longrightarrow \Lambda$ and a deformation of $I \in \mathcal{G}$, $\delta : \Gamma' \longrightarrow \mathcal{G}$ such that*

$$\psi(\tau) = \delta(\tau) * \varphi(\rho(\tau)), \forall \tau \in \Gamma'.$$

It is called miniversal if it has the minimal dimension of all the versal deformations.

Remark 2.4 *It is enough to compute a miniversal deformation of a point of the orbit; then, a miniversal deformation of any other point of the same orbit is induced from it by means of the group action.*

The “closed orbit lemma” ([12], p. 37) ensures that the referred starting point of Arnold's techniques holds in our case:

Proposition 2.5 *For all $a \in \mathcal{M}$, the orbit \mathcal{O}_a by the action of the algebraic group \mathcal{G} , is a submanifold of \mathcal{M} locally closed where the boundary is the union of orbits of strictly smaller dimension.*

Now, we recall the key relation between “versality ” and “transversality ”.

Definition 2.6 *Let $\mathcal{S} \subset \mathcal{V}$ be a submanifold of the manifold \mathcal{V} and $\varphi : \Lambda \longrightarrow \mathcal{V}$ be a differentiable map. We say that φ is transverse to \mathcal{S} in $\lambda \in \Lambda$ if $\varphi(\lambda) \in \mathcal{S}$ and the tangent space to \mathcal{V} in the point $\varphi(\lambda)$ verifies*

$$T_{\varphi(\lambda)}\mathcal{V} = \text{Im } d\varphi_\lambda + T_{\varphi(\lambda)}\mathcal{S}.$$

In particular, if Λ is a submanifold of \mathcal{V} (and φ is the inclusion), we say that it is transverse to \mathcal{S} in $\lambda \in \Lambda$ if

$$T_\lambda\mathcal{V} = T_\lambda\Lambda + T_\lambda\mathcal{S}.$$

We say that it is minitransverse if the sum is a direct sum.

As we have said above, the key point is the following proposition, proved in [1] for square matrices, and which can be generalized (for example [12]) to the cases like the ones here, where the equivalence classes are submanifolds given as orbits by the action of a Lie group.

Proposition 2.7 *A deformation $\varphi : \Lambda \longrightarrow \mathcal{M}$ of $a \in \mathcal{M}$ is versal/miniversal if and only if it is transverse/minitransverse to the orbit \mathcal{O}_a in the origin $O \in \Lambda$.*

Corollary 2.8 *A miniversal deformation of $a \in \mathcal{M}$ is determined by any supplementary subspace of $T_a\mathcal{O}_a$ in $T_a\mathcal{M} = \mathcal{M}$. Namely, if $\{e_1, e_2, \dots, e_r\}$ is a basis of a supplementary subspace of $T_a\mathcal{O}_a$ in \mathcal{M} , a miniversal deformation of $a \in \mathcal{M}$ is*

$$\varphi(\lambda_1, \lambda_2, \dots, \lambda_r) = a + \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_r e_r.$$

Moreover, r is the codimension of \mathcal{O}_a .

Finally, we recall the following result giving an explicit description of $T_a\mathcal{O}_a$:

Proposition 2.9 *Let the matrix $a \in \mathcal{M}$ and \mathcal{O}_a be its orbit by the action of the group \mathcal{G} ; then, the tangent space to this orbit in the point $a \in \mathcal{M}$ is*

$$T_a\mathcal{O}_a = \{[a, p] : p \in \mathcal{M}\},$$

where $[a, p] = ap - pa$.

Now, we are able to state and prove the main result of this section:

Theorem 2.10 *Let*

$$a = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathcal{M}.$$

Then, a miniversal deformation of this matrix is determined by the linear submanifold $a + N$, where N is the subspace formed by the matrices

$$x = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} \in \mathcal{M}$$

verifying the conditions

- (1) $A^*Z - ZB^* = 0$
- (2) $[A^*, X] - ZC^* = 0$
- (3) $[Y, B^*] - C^*Z = 0$

Proof. We consider the hermitien product in \mathcal{M} defined by

$$\langle x, p \rangle = \text{tr}(xp^*)$$

where $x = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$ and $p = \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix}$.

Because of (2.8), a miniversal deformation of a is given by $a + N$, where N is the orthogonal subspace of $T_a\mathcal{O}_a$.

So, a matrix $x \in \mathcal{M}$ will be in N if and only if

$$\langle x, [a, p] \rangle = 0, \quad \forall p \in \mathcal{M}.$$

Since

$$[a, p] = \begin{pmatrix} AP - PA & AR + CQ - PC - RB \\ 0 & BQ - QB \end{pmatrix},$$

this condition is equivalent to

$$\text{tr}(XP^*A^* - XA^*P^* + ZR^*A^* + ZQ^*C^* - ZC^*P^* - ZB^*R^*) + \text{tr}(YQ^*B^* - YB^*Q^*) = 0$$

$$\forall p \in \mathcal{M}.$$

Then, because of the invariance of the trace by the circular permutations, the last condition is equivalent to

$$\text{tr}(A^*XP^* - XA^*P^* + A^*ZR^* - ZC^*P^* - ZB^*R^*) + \text{tr}(C^*ZQ^* + B^*YQ^* - YB^*Q^*) = 0$$

$$\forall p \in \mathcal{M}.$$

Getting the common factors out, it becomes

$$\text{tr}((A^*X - XA^* - ZC^*)P^* + (A^*Z - ZB^*)R^*) + \text{tr}((C^*Z + B^*Y - YB^*)Q^*) = 0$$

$$\forall P \in M_n(C), Q \in M_m(C), R \in M_{n,m}(C),$$

which is equivalent to

$$\left\langle \begin{pmatrix} A^*X - XA^* - ZC^* & A^*Z - ZB^* \\ 0 & C^*Z + B^*Y - YB^* \end{pmatrix}, \begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} \right\rangle = 0$$

$$\forall p \in \mathcal{M}.$$

Hence, $x \in N$ if and only if the first matrix is zero. ■

3 Obtention of Miniversal Deformations of Marked Matrices

We recall that if f is an endomorphism of a finite dimensional vector space \mathcal{X} , an f -invariant subspace F of \mathcal{X} is said to be marked if there is a Jordan basis of F that can be extended to a Jordan basis of \mathcal{X} with regard to f .

Definition 3.1 *We say that $a \in \mathcal{M}$ is a marked matrix if $C^n \times \{0\}$ is an a -invariant marked subspace of C^{n+m} . Notice that if A and B are nilpotent Jordan matrices; this means that there is a permutation matrix $p \in \mathcal{G}$ such that $p^{-1}ap$ is a nilpotent Jordan matrix.*

As we have said in the introduction, we will solve the equations in (2.10) explicitly in those cases when a is a nilpotent marked matrix. Because of (2.4), it is sufficient to obtain the versal deformation of any matrix in this orbit. It is easily see that any nilpotent marked matrix is equivalent to a matrix of the form described in the following definition:

Definition 3.2 *We say that a marked nilpotent matrix $a \in \mathcal{M}$ is in canonical form if*

- (1) $A = \text{diag}(A_1, \dots, A_r)$, where A_1, \dots, A_r are nilpotent matrices in Jordan form of sizes $\gamma_1, \dots, \gamma_r$ respectively, and $\gamma_1 + \dots + \gamma_r = n$.
- (2) $B = \text{diag}(B_1, \dots, B_s)$, where B_1, \dots, B_s are nilpotent matrices in Jordan form of sizes β_1, \dots, β_s respectively, and $\beta_1 + \dots + \beta_s = m$.
- (3) $C = [C_{i,j}]_{1 \leq i \leq r, 1 \leq j \leq s}$ $C_{i,j} \in M_{\gamma_i, \beta_j}(C)$ such that

$$C_{ii} = \left[\begin{array}{c|c} 0 \dots 0 & 1 \\ \hline \mathbf{0} & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \end{array} \right] \quad \text{if } 1 \leq i \leq \rho \leq \min(r, s)$$

and $C_{ij} = 0$ for any other cases.

- (4) $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_\rho$.
- (5) $\beta_i \geq \beta_{i+1}$ if $\begin{cases} 0 < i < \rho \\ \gamma_i = \gamma_{i+1} \end{cases}$
- (6) $\gamma_{\rho+1} \geq \gamma_{\rho+2} \geq \dots \geq \gamma_r$.
- (7) $\beta_{\rho+1} \geq \beta_{\rho+2} \geq \dots \geq \beta_s$.

We say that the matrix $a = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is of type $(\tilde{\gamma}, \tilde{\beta}, \rho)$, where $\tilde{\gamma} = (\gamma_1, \dots, \gamma_r)$ and $\tilde{\beta} = (\beta_1, \dots, \beta_s)$

Observe that ρ is the number of chains in a Jordan basis of $C^n \times \{0\}$ with regard to A that can be extended to chains of a Jordan basis of C^{n+m} with regard to a .

Also notice that $\tilde{\gamma}$ and $\tilde{\beta}$ have the same elements as γ and β , but they are not in non increasing order.

Example 3.3 The next matrix a is a marked nilpotent matrix in canonical form of type $((3, 2, 1, 3, 1), (2, 4, 1, 3, 2), 3)$:

$$\begin{bmatrix} 1 & & & & & & 1 & & & & \\ & 1 & & & & & & & & & \\ & & 1 & & & & & & 1 & & \\ & & & 1 & & & & & & 1 & \\ & & & & 1 & & & & & & 1 \\ & & & & & 1 & & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & 1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & 1 & \\ & & & & & & & & & & 1 \end{bmatrix}$$

Then, the Segre characteristic of a is $(5, 6, 2, 3, 3, 2, 1)$.

When we solve the set of equations in (2.10), the following special type of Toeplitz matrices will often appear:

- Definition 3.4** (1) We say that a matrix $X = (x_{i,j}) \in M_{\gamma,\beta}(C)$ is a T-matrix if it is a Toeplitz matrix; that is to say, if it is constant along the diagonals.
- (2) If X is a T-matrix such that all the diagonals from the $(\lambda + 1)$ th are zero (beginning to count from the right upper corner), we say that X is a λ T-matrix.
- (3) If X is a λ T-matrix, where $\lambda = \min(\gamma, \beta)$, we simply say that X is a UTT-matrix (upper triangular Toeplitz matrix).

For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$$

are 4T, 3T and 3T matrices, respectively, and

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

are UTT-matrices.

Definition 3.5 We say that a block matrix

$$X = [X_{i,j}]_{1 \leq i \leq r, 1 \leq j \leq s} \quad X_{i,j} \in M_{\gamma_i, \beta_j}(C)$$

is a block T-matrix if each block $X_{i,j}$ is a T-matrix. We define a block UTT-matrix analogously.

We are now going to solve equations (1), (2) and (3) in (2.10) when the matrix $a \in \mathcal{M}$ is a marked nilpotent matrix in canonical form.

Lemma 3.6 Let $M \in M_\gamma(\mathbb{C})$ and $N \in M_\beta(\mathbb{C})$ be Jordan nilpotent non derogatory matrices. Then, a matrix $Z \in M_{\gamma, \beta}(\mathbb{C})$ verifies the equation

$$(1) \quad M^*Z - ZN^* = 0$$

if and only if Z is a UTT-matrix.

Proof. It is clear that $Z = (z_{i,j})$ verifies

$$\begin{bmatrix} z_{2,1} & z_{2,2} & \dots & z_{2,\beta} \\ z_{3,1} & z_{3,2} & \dots & z_{3,\beta} \\ \vdots & \vdots & & \vdots \\ z_{\gamma,1} & z_{\gamma,2} & \dots & z_{\gamma,\beta} \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & z_{1,1} & z_{1,2} & \dots & z_{1,\beta-1} \\ 0 & z_{2,1} & z_{2,2} & \dots & z_{2,\beta-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & z_{\gamma,1} & z_{\gamma,2} & \dots & z_{\gamma,\beta-1} \end{bmatrix},$$

which is equivalent to $z_{h,\ell} = z_{h+1,\ell+1}$. So, Z is a T-matrix, and being $z_{h,1} = z_{\gamma,\beta-\ell} = 0$ if $h, \ell > 1$, we conclude that Z is a UTT-matrix. \blacksquare

In order to solve equations (1), (2) and (3) in (2.10), we decompose X, Y, Z into blocks:

$$X = [X_{i,j}]_{1 \leq i,j \leq r} \quad X_{i,j} \in M_{\gamma_i, \gamma_j}(C)$$

$$Y = [Y_{t,k}]_{1 \leq t,k \leq s} \quad Y_{t,k} \in M_{\beta_t, \beta_k}(C)$$

$$Z = [Z_{i,k}]_{1 \leq i \leq r, 1 \leq k \leq s} \quad Z_{i,k} \in M_{\gamma_i, \beta_k}(C).$$

The next lemma follows immediately from (3.2):

Lemma 3.7 With the notation in (3.2) the equations (1), (2) and (3) in (2.10) are equivalent to the following ones:

- (1) $A_i^* Z_{ik} - Z_{ik} B_k^* = 0, \quad 1 \leq i \leq r, \quad 1 \leq k \leq s$
- (2) $A_i^* X_{ij} - X_{ij} A_j^* = Z_{ij} C_{jj}^*, \quad 1 \leq i \leq r, \quad 1 \leq j \leq \rho$
- (3) $A_i^* X_{ij} - X_{ij} A_j^* = 0, \quad 1 \leq i \leq r, \quad \rho < j \leq r$
- (4) $Y_{tk} B_k^* - B_t^* Y_{tk} = C_{tt}^* Z_{tk}, \quad 1 \leq t \leq \rho, \quad 1 \leq k \leq s$
- (5) $Y_{tk} B_k^* - B_t^* Y_{tk} = 0, \quad \rho < t \leq s, \quad 1 \leq k \leq s.$

Consequently, in order to obtain the solution of the above set of equations, we are led to consider the following four sets of equations:

(I) If $1 \leq i, j \leq \rho$

$$\begin{aligned} 1.I \quad & A_i^* Z_{ij} - Z_{ij} B_j^* = 0. \\ 2.I \quad & A_i^* X_{ij} - X_{ij} A_j^* = Z_{ij} C_{jj}^*. \\ 3.I \quad & Y_{ij} B_j^* - B_i^* Y_{ij} = C_{ii}^* Z_{ij}. \end{aligned}$$

(II) If $i > \rho, j \leq \rho$

$$\begin{aligned} 1.II \quad & A_i^* Z_{ij} - Z_{ij} B_j^* = 0. \\ 2.II \quad & A_i^* X_{ij} - X_{ij} A_j^* = Z_{ij} C_{jj}^*. \\ 3.II \quad & Y_{ij} B_j^* - B_i^* Y_{ij} = 0. \end{aligned}$$

(III) If $i \leq \rho, j > \rho$

$$\begin{aligned} 1.III \quad & A_i^* Z_{ij} - Z_{ij} B_j^* = 0. \\ 2.III \quad & A_i^* X_{ij} - X_{ij} A_j^* = 0. \\ 3.III \quad & Y_{ij} B_j^* - B_i^* Y_{ij} = C_{ii}^* Z_{ij}. \end{aligned}$$

(IV) If $i, j > \rho$

$$\begin{aligned} 1.IV \quad & A_i^* Z_{ij} - Z_{ij} B_j^* = 0. \\ 2.IV \quad & A_i^* X_{ij} - X_{ij} A_j^* = 0. \\ 3.IV \quad & Y_{ij} B_j^* - B_i^* Y_{ij} = 0. \end{aligned}$$

The following theorem describes the corresponding solutions:

Theorem 3.8 (First Miniversal Deformation) *Let $a \in \mathcal{M}$ be a marked nilpotent matrix in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$. Then, a miniversal deformation of $a \in \mathcal{M}$ is $a + N$ where N is the subspace formed by the matrices $x = \begin{pmatrix} X & Z \\ O & Y \end{pmatrix}$ such that*

(I) $1 \leq i, j \leq \rho$

a) If $\gamma_i \leq \gamma_j$ or $\beta_i \geq \beta_j$

$Z_{ij} = 0$ and X_{ij}, Y_{ij} are UTT-matrices.

b) If $\gamma_i > \gamma_j$ and $\beta_i < \beta_j$

Z_{ij} are μ_{ij} -T-matrices where $\mu_{ij} = \min(\gamma_i - \gamma_j, \beta_j - \beta_i)$.

X_{ij} are $(\gamma_j + \mu_{ij})$ -T-matrices and the diagonals

$\gamma_j + 1, \dots, \gamma_j + \mu_{ij}$ are equal to the diagonals $1, \dots, \mu_{ij}$ of Z_{ij} .

Y_{ij} are $(\beta_i + \mu_{ij})$ -T-matrices and the diagonals

$\beta_i + 1, \dots, \beta_i + \mu_{ij}$ are equal to the diagonals $1, \dots, \mu_{ij}$ of Z_{ij} .

(II) $i > \rho, j \leq \rho$

a) If $\gamma_i \leq \gamma_j$

$Z_{ij} = 0$ and X_{ij}, Y_{ij} are UTT-matrices.

b) If $\gamma_i > \gamma_j$

Z_{ij} are δ_{ij} T-matrices where $\delta_{ij} = \min(\gamma_i - \gamma_j, \beta_j)$.

X_{ij} are $(\gamma_j + \delta_{ij})$ T-matrices and the diagonals $\gamma_j + 1, \dots, \gamma_j + \delta_{ij}$ are equal to the diagonals $1, \dots, \delta_{ij}$ of Z_{ij} .

Y_{ij} are UTT-matrices.

(III) $i \leq \rho, j > \rho$

a) If $\beta_i \geq \beta_j$

$Z_{ij} = 0$ and X_{ij}, Y_{ij} are UTT-matrices.

b) If $\beta_i < \beta_j$

Z_{ij} are ε_{ij} T-matrices where $\varepsilon_{ij} = \min(\gamma_i, \beta_j - \beta_i)$.

X_{ij} are UTT-matrices.

Y_{ij} are $(\beta_i + \varepsilon_{ij})$ T-matrices and the diagonals $\beta_i + 1, \dots, \beta_i + \varepsilon_{ij}$ are equal to the diagonals $1, \dots, \varepsilon_{ij}$ of Z_{ij} .

(IV) $i, j > \rho$

X_{ij}, Y_{ij}, Z_{ij} are UTT-matrices.

Proof. From lemma 3.7, Z_{ij} is a solution of (1.I) if it is a UTT-matrix. The remaining equations are then transformed into the following ones

$$(2.I) \quad \begin{bmatrix} x_{2,1} & x_{2,2} & \dots & x_{2,\gamma_j} \\ \vdots & & & \vdots \\ x_{\gamma_i,1} & x_{\gamma_i,2} & \dots & x_{\gamma_i,\gamma_j} \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} z_{1,\beta_j} & x_{1,1} & \dots & x_{1,\gamma_j-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ z_{\gamma_i,\beta_j} & x_{\gamma_i,1} & \dots & x_{\gamma_i,\gamma_j-1} \end{bmatrix}$$

and

$$(3.I) \quad \begin{bmatrix} 0 & y_{1,1} & \dots & y_{1,\beta_j-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & y_{\beta_i,1} & \dots & y_{\beta_i,\beta_j-1} \end{bmatrix} = \begin{bmatrix} y_{2,1} & \dots & \dots & y_{2,\beta_j} \\ \vdots & & & \vdots \\ y_{\beta_i,1} & \dots & \dots & y_{\beta_i,\beta_j} \\ z_{1,1} & \dots & \dots & z_{1,\beta_j} \end{bmatrix}$$

where the matrices Z_{ij} being UTT-matrices, are such that

$$\begin{cases} z_{1,\beta_j-h+1} = z_{h,\beta_j} \\ z_{h,\beta_j} = 0 \quad \text{si } h > \min(\gamma_i, \beta_j) \end{cases}$$

So, from the above equations we have that

$$(2.I)' \quad \begin{cases} x_{h,\ell} = x_{h+1,\ell+1} \\ x_{\gamma_i,\ell} = 0 \\ x_{h+1,1} = z_{h,\beta_j} \\ z_{\gamma_i,\beta_j} = 0 \end{cases} \quad \begin{cases} \text{if } 1 \leq \ell < \gamma_j \\ \text{if } 1 \leq h < \gamma_i \end{cases}$$

$$(3.I)' \quad \begin{cases} y_{h,\ell} = y_{h+1,\ell+1} \\ y_{h,1} = 0 & \text{if } 1 < h \leq \beta_i \\ y_{\beta_i,\ell} = z_{1,\ell+1} & \text{if } 1 \leq \ell < \beta_j \\ z_{1,1} = 0 \end{cases}$$

From the first equation of (2.I)' and (3.I)', we conclude that X_{ij} i Y_{ij} are T-matrices.

Two cases are to be considered:

- a) If $\gamma_i \leq \gamma_j$ ($\beta_i \geq \beta_j$), the second equation of (2.I)' ((3.I)') says that X_{ij} (Y_{ij}) is a UTT-matrix. In this case, from the remaining equations of each group we see that the last column and the first row of Z_{ij} are zero. Hence, $Z_{ij} = 0$. Then taking into account the third equation of each group, we obtain that X_{ij} and Y_{ij} are UTT-matrices.
- b) If $\gamma_i > \gamma_j$ and $\beta_i < \beta_j$, taking into account that X_{ij} and Y_{ij} are T-matrices, we see, from the second equation of each group, that

$$\begin{cases} x_{h+1,1} = 0 & \text{if } \gamma_i - \gamma_j < h < \gamma_i \\ y_{\beta_i,\beta_j-h} = 0 & \text{if } \beta_j - \beta_i < h < \beta_j \end{cases}$$

and taking also into account the remaining equations, we conclude that

$$\begin{cases} z_{h,\beta_j} = 0 & \text{if } \gamma_i - \gamma_j < h \leq \gamma_i \\ z_{1,\beta_j-h+1} = 0 & \text{if } \beta_j - \beta_i < h \leq \beta_j \end{cases}$$

thus Z_{ij} is a UTT-matrix with

$$z_{h,\beta_j} = 0 \quad \text{si } h > \min(\gamma_i - \gamma_j, \beta_j - \beta_i)$$

and

$$\begin{cases} x_{h+1,1} = z_{h,\beta_j} & \text{si } 1 \leq h \leq \gamma_i - \gamma_j \\ y_{\beta_i,\beta_j-h} = z_{1,\beta_j-h+1} & \text{si } 1 \leq h \leq \beta_j - \beta_i \end{cases}$$

In summary, the solutions of (I) have the following form:

$$X_{ij} = \left[\begin{array}{cccc} x_{\gamma_j,\gamma_j} & \dots & x_{2,\gamma_j} & x_{1,\gamma_j} \\ z_{1,\beta_j} & \ddots & \ddots & x_{2,\gamma_j} \\ \vdots & \ddots & \ddots & \vdots \\ z_{\mu,\beta_j} & \ddots & \ddots & x_{\gamma_j,\gamma_j} \\ 0 & \ddots & \ddots & z_{1,\beta_j} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & z_{\mu,\beta_j} \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{array} \right] \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}} \right\} \gamma_i - \gamma_j - \mu$$

where $\mu = \min(\gamma_i - \gamma_j, \beta_j - \beta_i)$

$$Y_{ij} = \begin{bmatrix} \overbrace{0 \ \dots \ 0}^{\beta_j - \beta_i - \mu} & z_{\mu, \beta_j} & \dots & z_{1, \beta_j} & y_{\beta_i, \beta_j} & \dots & y_{2, \beta_j} & y_{1, \beta_j} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & y_{2, \beta_j} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & z_{\mu, \beta_j} & \dots & z_{1, \beta_j} & y_{\beta_i, \beta_j} \end{bmatrix}$$

$$Z_{ij} = \begin{bmatrix} \overbrace{0 \ \dots \ 0}^{\beta_j - \mu} & z_{\mu, \beta_j} & \dots & z_{2, \beta_j} & z_{1, \beta_j} \\ \vdots & \ddots & \ddots & \ddots & z_{2, \beta_j} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & \dots & 0 & z_{\mu, \beta_j} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \end{bmatrix}$$

and (I) is proved.

The proof of the remaining cases is similar and it is omitted. ■

As an application, we obtain the dimension of \mathcal{O}_a (see (2.8)):

Corollary 3.9 *Let $a \in \mathcal{M}$ be a marked nilpotent matrix in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$. Then, the codimension of the orbit of $a \in \mathcal{M}$ is:*

$$\begin{aligned} \text{codim} \mathcal{O}_a &= \\ &= \sum_{1 \leq i, j \leq r} \min(\gamma_i, \gamma_j) + \sum_{1 \leq i, j \leq s} \min(\beta_i, \beta_j) + \sum_{1 \leq i, j \leq p} \max[0, \min(\gamma_i - \gamma_j, \beta_j - \beta_i)] + \\ &+ \sum_{\substack{p < i \leq r \\ 1 \leq j \leq p}} \max[0, \min(\gamma_i - \gamma_j, \beta_j)] + \sum_{\substack{1 \leq i \leq p \\ p < j \leq s}} \max[0, \min(\gamma_i, \beta_j - \beta_i)] + \sum_{\substack{p < i \leq r \\ p < j \leq s}} \min(\gamma_i, \beta_j) \end{aligned}$$

Remark 3.10 *Notice that $\text{codim} \mathcal{O}_a > 0$. Hence, none marked matrix is structurally stable.*

[illegible]

We will now derive a new miniversal deformation of $a \in \mathcal{M}$ where there are no repeated parameters. We will construct it by taking an appropriate basis of a suitable supplementary of $T_a\mathcal{O}_a$.

$$\begin{array}{lll} a_{ij}^k \in \mathcal{M} & 1 \leq i, j \leq r & 1 \leq k \leq \min(\gamma_i, \gamma_j) \\ b_{ij}^k \in \mathcal{M} & 1 \leq i, j \leq s & 1 \leq k \leq \min(\beta_i, \beta_j) \\ c_{ij}^k \in \mathcal{M} & 1 \leq i \leq r, 1 \leq j \leq s & 1 \leq k \leq \min(\gamma_i, \beta_j) \end{array}$$

as the matrices having the same block sizes as in $a \in \mathcal{M}$, where all the entries are 0 except one 1 placed in the first row of the block A_{ij} , B_{ij} or C_{ij} respectively, and in their k -column (ordering the columns from right to left).

Let $a \in \mathcal{M}$ be a marked nilpotent matrix in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$. In order to simplify the notation, we write:

$$\begin{aligned}\gamma' &= \sum_{1 \leq i, j \leq r} \min(\gamma_i, \gamma_j) \\ \beta' &= \sum_{1 \leq i, j \leq s} \min(\beta_i, \beta_j) \\ \mu &= \sum_{1 \leq i, j \leq p} \max[0, \min(\gamma_i - \gamma_j, \beta_j - \beta_i)] + \sum_{\substack{p < i \leq r \\ 1 \leq j \leq p}} \max[0, \min(\gamma_i - \gamma_j, \beta_j)] + \\ &\quad + \sum_{\substack{1 \leq i \leq p \\ p < j \leq s}} \max[0, \min(\gamma_i, \beta_j - \beta_i)] + \sum_{\substack{p < i \leq r \\ p < j \leq s}} \min(\gamma_i, \beta_j)\end{aligned}$$

Therefore, with this notation, we have $\text{codim } \mathcal{O}_a = \gamma' + \beta' + \mu$, and

$$(a) \quad \hat{x} = (x_{ij}^k) \in \mathbb{C}^{\gamma'} \quad \left\{ \begin{array}{l} 1 \leq i, j \leq r \\ 1 \leq k \leq \min(\gamma_i, \gamma_j) \end{array} \right\}$$

$$(b) \quad \hat{y} = (y_{ij}^k) \in \mathbb{C}^{\beta'} \quad \left\{ \begin{array}{l} 1 \leq i, j \leq s \\ 1 \leq k \leq \min(\beta_i, \beta_j) \end{array} \right\}$$

$$(c) \quad \hat{z} = (z_{ij}^k) \in \mathbb{C}^{\mu} \quad \left\{ \begin{array}{l} 1 \leq i, j \leq p, 0 < k \leq \min(\gamma_i - \gamma_j, \beta_j - \beta_i) \\ p < i \leq r, 1 \leq j \leq p, 0 < k \leq \min(\gamma_i - \gamma_j, \beta_j) \\ 1 < i \leq p, p < j \leq s, 0 < k \leq \min(\gamma_i, \beta_j - \beta_i) \\ p < i \leq r, p < j \leq s, 1 \leq k \leq \min(\gamma_i, \beta_j) \end{array} \right\}$$

We denote by S_a the vector space spanned by the matrices $a_{ij}^k, b_{ij}^k, c_{ij}^k$, where the i, j, k indices vary in the index sets in (a), (b) and (c), respectively.

Theorem 3.13 (Second Miniversal Deformation) *Let $a \in \mathcal{M}$ be a marked nilpotent matrix in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$. Then, a miniversal deformation of $a \in \mathcal{M}$ is the map*

$$\begin{aligned}\varphi : \mathbb{C}^{\gamma'} \times \mathbb{C}^{\beta'} \times \mathbb{C}^{\mu} &\longrightarrow \mathcal{M} \\ (\hat{x}, \hat{y}, \hat{z}) &\longmapsto a + \sum_{i,j,k} x_{ij}^k a_{ij}^k + \sum_{i,j,k} y_{ij}^k b_{ij}^k + \sum_{i,j,k} z_{ij}^k c_{ij}^k\end{aligned}$$

Proof. By construction, the set of $\{a_{ij}^k, b_{ij}^k, c_{ij}^k\}_{i,j,k}$ matrices is linearly independent, so that the dimension of the subspace S_a spanned by them is $\gamma' + \beta' + \mu$, which is also the dimension of the orthogonal of $T_a \mathcal{O}_a$ in accordance with the corollary (3.9). We will see that S_a is a supplementary subspace of $T_a \mathcal{O}_a$ by proving that its intersection is the null space; in order to do so, we will prove that for every non null vector of S_a there is a vector of $(T_a \mathcal{O}_a)^\perp$ such that their product is not zero.

Notice that if $x \in (T_a \mathcal{O}_a)^\perp$; then we have

$$\begin{aligned}
 \langle x, a_{ij}^k \rangle &= \text{tr} \left[X_{ij} \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \middle| \begin{array}{c} \mathbf{0} \end{array} \right) \right]_{\gamma_j - k + 1} = (X_{ij})_{1, \gamma_j - k + 1} \\
 \langle x, b_{ij}^k \rangle &= (Y_{ij})_{1, \beta_j - k + 1} \\
 \langle x, c_{ij}^k \rangle &= (Z_{ij})_{1, \beta_j - k + 1}
 \end{aligned}$$

where $(X_{ij})_{h,l}$ is the h -row l -column entry of the X_{ij} matrix.

Now, let

$$v = \sum_{i,j,k} x_{ij}^k a_{ij}^k + \sum_{i,j,k} y_{ij}^k b_{ij}^k + \sum_{i,j,k} z_{ij}^k c_{ij}^k$$

be a vector of S_a . We consider the vector $x = \left(\begin{array}{c|c} X & Z \\ \hline \mathbf{0} & Y \end{array} \right) \in (T_a \mathcal{O}_a)^\perp$ defined by

$$\begin{cases} (X_{ij})_{1, \gamma_j - k + 1} = \overline{x_{ij}^k} \\ (Y_{ij})_{1, \beta_j - k + 1} = \overline{y_{ij}^k} \\ (Z_{ij})_{1, \beta_j - k + 1} = \overline{z_{ij}^k} \end{cases}$$

where the indices vary in the sets in (a), (b) and (c) respectively.

Then,

$$\langle v, x \rangle = \sum_{i,j,k} |x_{ij}^k|^2 + \sum_{i,j,k} |y_{ij}^k|^2 + \sum_{i,j,k} |z_{ij}^k|^2$$

and this implies that $\langle v, x \rangle = 0$ if and only if $v = 0$. ■

Example 3.14 The new miniversal deformation in example (3.11) is

$$\left[\begin{array}{cccc|cccc|c|cc|cc|cc} * & * & * & * & * & * & * & * & * & 1 & & * & & * & \\ 1 & & & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & & & \\ * & * & * & * & * & & * & * & * & & & 1 & & & \\ & & 1 & & & & & & & & & & & & \\ * & & * & * & * & & * & * & * & & & 1 & & * & * \\ * & * & * & * & * & * & * & * & * & & & * & * & * & * \\ & & & & 1 & & & & & & & & & & \\ & & & & 1 & & & & & & & & & & \\ * & & * & * & * & & * & * & & & & & & * & * \\ \hline & & & & & & * & * & & * & * & * & * & * & * \\ & & & & & & 1 & & & & & & & & \\ & & & & & & * & * & * & * & * & * & * & * & \\ & & & & & & & 1 & & & & & & & \\ & & & & & & & & 1 & & & & & & \\ & & & & & & & & & 1 & & & & & \\ & & & & & & * & & * & * & & * & * & * & \\ & & & & & & * & * & * & * & * & * & * & * & \\ & & & & & & & & & 1 & & & & & \\ & & & & & & & & & 1 & & & & & \\ & & & & & & * & & * & * & * & * & * & * & \\ & & & & & & & & & & 1 & & & & \end{array} \right]$$

4 Relation with the Carlson Problem

We recall that the Carlson problem asks about the Jordan invariants of $a \in \mathcal{M}$ when A, B, C vary in such a way that the Jordan invariants of the restriction block and the quotient block are preserved. It is well-known that the problem can be reduced to the nilpotent case (and even to the particular case when A and B are Jordan matrices); this means that only the Segre characteristics are involved. Hence, we define:

Definition 4.1 Let α, γ, β be three partitions with $|\alpha| = n$, $|\gamma| = d$, $|\beta| = n - d$. We say that α is Carlson-compatible with (γ, β) (or that the triple (α, γ, β) is Carlson-compatible) if there is a nilpotent matrix $a = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ such that the Segre characteristics of a, A, B are α, γ, β respectively. Then we say that a is a Carlson-realization of (γ, β) or, more precisely, of the triple (α, γ, β) .

For example, the marked matrices in (3.2) are Carlson-realizations of (γ, β) .

In general, the matrices in (3.13) do not preserve the invariants of the initial one. However, because of Arnold's deformations of a square matrix (see [1]), γ and β are preserved if and only if $X = 0$ and $Y = 0$, respectively. Hence, the miniversal deformation in (3.13) gives a representation of the Carlson-realizations of (γ, β) near the initial matrix. More precisely:

Proposition 4.2 Let $a \in \mathcal{M}$ be a marked nilpotent matrix in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$ and φ its miniversal deformation in (3.13). Given any deformation of $a \in \mathcal{M}$, $\psi : \Gamma \rightarrow \mathcal{M}$,

such that $\psi(\tau)$ is a Carlson-realization of (γ, β) for any $\tau \in \Gamma$, there is a neighbourhood of the origin $\Gamma' \subset \Gamma$, a differentiable map $\rho_3 : \Gamma' \longrightarrow C^\mu$ and a deformation of $I \in \mathcal{G}$, $\delta : \Gamma' \longrightarrow \mathcal{G}$ such that

$$\psi(\tau) = (\delta(\tau))^{-1} \varphi(0, 0, \rho_3(\tau)) \delta(\tau).$$

Proof. In general, for any deformation of $a \in \mathcal{M}$, there are $\hat{x} = \rho_1(\tau)$, $\hat{y} = \rho_2(\tau)$, $\hat{z} = \rho_3(\tau)$ such that

$$\psi(\tau) = (\delta(\tau))^{-1} \varphi(\rho_1(\tau), \rho_2(\tau), \rho_3(\tau)) \delta(\tau).$$

If $\psi(\tau)$ is a realization of (γ, β) , the restriction block and the quotient block in $\varphi(\rho_1(\tau), \rho_2(\tau), \rho_3(\tau))$ must have Segre characteristic γ and β respectively. As we have commented above, this is only possible if $\rho_1(\tau) = 0$ and $\rho_2(\tau) = 0$. ■

In particular, we conclude with the following result:

Corollary 4.3 *Let a be a marked nilpotent matrix in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$. If a is stable by the deformations that preserve the Segre characteristics of the restriction and the quotient, then*

- (i) $\rho = \min(r, s)$.
- (ii) $\gamma_i \geq \gamma_{i+1}$.
- (iii) $\beta_i \geq \beta_{i+1}$.

Proof. The number of parameters in Z must be zero, that is to say $\mu = 0$. ■

More generally, let us see that any Carlson-compatible partition with (γ, β) appears in the miniversal deformation (3.13) of a matrix a of type $(\gamma, \beta, 0)$ by taking $X = 0$ and $Y = 0$ (Notice that the above matrix a is a trivial Carlson-realization of the triple $(\gamma \cup \beta, \gamma, \beta)$). This representation improves the well known "condensation lemma" which asserts that any Carlson-compatible partition with a given pair (γ, β) can be realized by means of a matrix $\begin{pmatrix} N(\gamma) & C \\ 0 & N(\beta) \end{pmatrix}$, where $N(\gamma)$, $N(\beta)$ are nilpotent Jordan matrices having Segre characteristic γ, β , respectively; the only non zero entries in C are the ones placed in the rows which correspond to null rows in $N(\gamma)$. Here we prove that several of these entries in C can be assumed to be zero.

Theorem 4.4 *Given a pair of partitions (γ, β) , realizations of all the Carlson-compatible partitions with them are obtained by considering the miniversal deformation (3.13) for $p = 0$, and taking $X = 0$ and $Y = 0$.*

In particular, they are of the form $\begin{pmatrix} N(\gamma) & Z \\ 0 & N(\beta) \end{pmatrix}$, where $N(\gamma)$, $N(\beta)$ are nilpotent Jordan matrices having Segre characteristic γ, β respectively; the only non zero entries in Z are some of the ones placed in the rows which correspond to null rows in $N(\gamma)$. In addition, the parameters in Z can be taken as small as desired.

Proof. In [4] it is shown that realizations of all Carlson-compatible partitions with (γ, β) occur in any neighbourhood of the marked ones. Hence, all of them appear in the set of the miniversal

deformations (3.13) when all possible types $(\tilde{\gamma}, \tilde{\beta}, p)$ are considered for fixed (γ, β) . Finally, notice that all these nilpotent marked matrices in canonical form of type $(\tilde{\gamma}, \tilde{\beta}, p)$ appear in the miniversal deformation of the one of type $(\gamma, \beta, 0)$; it is sufficient to take all the entries valued 0, except some z_{ij}^1 , in such a way that there is at least a non zero entry for every i and for every j . ■

Remark 4.5 *In particular it follows that there are realizations of all the Carlson-compatible partitions with (γ, β) in any neighbourhood of the trivial one $a = \text{diag}(N(\gamma), N(\beta))$. Notice that, according to the notation in (3.2), this matrix is a marked nilpotent matrix in canonical form of type $(\gamma, \beta, 0)$.*

Example 4.6 Let $\gamma = (3, 3, 2, 1, 1)$ and $\beta = (4, 3, 2, 2, 1)$ be the Segre characteristics of example (3.3). Then, we have that $\gamma \cup \beta = (4, 3, 3, 3, 2, 2, 2, 1, 1, 1)$.

The family of deformations that preserves the pair (γ, β) and that has matricial realizations of all the compatible Littlewood-Richardson sequences is

1					*	*		*	*	*	*	*	*	*	*	*
1					*	*		*	*	*	*	*	*	*	*	*
	1				*	*		*	*	*	*	*	*	*	*	*
					*			*				*	*	*	*	*
			1					*	*	*	*	*	*	*	*	*
			1		*			*	*		*	*	*	*	*	*
					1											
						1										
							1									
								1								

If we deform the first case preserving the pair (γ, β) , we obtain:

$$\left[\begin{array}{cc|c|cc|c} 0 & 0 & 0 & x & y & z \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & t & u \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

which gives us the following Segre characteristic depending on the parameter's values:

Segre char.	$yu - zt$	y	x	z	t	u
(4,2)	$\neq 0$	$\neq 0$				
(3,3)	$\neq 0$	0				
(4,1,1)	0	$\neq 0$				
(3,2,1)	0	0	$\neq 0$	(*)		
(3,2,1)	0	0	0	(**)		
(2,2,2)	0	0	0	0	0	$\neq 0$
(3,1,1,1)	0	0	$\neq 0$	0	0	0
(2,2,1,1)	0	0	0	0	0	0

(*) $z \neq 0 \circ t \neq 0 \circ u \neq 0$.

(**) $z \neq 0 \circ t \neq 0$.

Therefore, they are realizations of all the Carlson-compatible partitions with $\gamma = \beta = (2, 1)$ (see (4.5)). In particular, the above marked ones are included, and the seventh is not marked. Moreover, such deformation gives the corresponding Littlewood-Richardson sequences of the partitions (2,1) and (2,1) (notice that partition (2,1) is auto-conjugate):

$$\begin{array}{lll} (2,1) & (2,2,1) & (2,2,1,1)=(4,2)^* \\ (2,1) & (2,2,1) & (2,2,2)=(3,3)^* \\ (2,1) & (3,1,1) & (3,1,1,1)=(4,1,1)^* \\ (2,1) & (3,2) & (3,2,1)=(3,2,1)^* \\ (2,1) & (3,1,1) & (3,2,1)=(3,2,1)^* \\ (2,1) & (3,2) & (3,3)=(2,2,2)^* \\ (2,1) & (4,1) & (4,1,1)=(3,1,1,1)^* \\ (2,1) & (4,1) & (4,2)=(2,2,1,1)^* \end{array}$$

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